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Fundamental progroupoid and bundles with a structural category ☆

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Abstract

In this paper, for a given space X, a structural category $\mathcal{C}$, and a faithful functor η from $\mathcal{C}$ to the category of spaces, we introduce a notion of $(\mathcal{C},\eta)$-bundle which contains as particular cases, the notions of covering space, of overlaying space (introduced by Fox), of suspension foliation and other well-known topological structures.

The new notion allows us to use sheaf theory and category theory in order to obtain some classification theorems which appear in terms of equivalences of categories. We prove that the category $(\mathcal{C},\eta)$-bundle(X) of $(\mathcal{C},\eta)$-bundles over X is equivalent to the category pro$(\pi CX,\mathcal{C})$, which is determined by the fundamental groupoid of X and the structural category $\mathcal{C}$. As particular cases we obtain the standard classification of covering spaces, Fox's classification theorem for overlays with a finite number of leaves and the standard classification of suspension foliations.

This paper illustrates the importance of the fundamental progroupoid of a space X, which plays in shape theory the role of the standard fundamental groupoid. If the space X satisfies some additional properties, the progroupoid πCX can be reduced to a surjective progroup, a groupoid or a group. In some cases a surjective progroupoid can be replaced by a topological prodiscrete group. In all these cases the category pro$(\pi CX,\mathcal{C})$ also reduces to well-known categories.

Keywords: Fundamental progroupoid; $(\mathcal{C},\eta)$-bundle; Flat bundle; Locally constant presheaf; Covering projection; Overlay

AMS classification: 55Q07; 55U40; 57M10; 18B25; 18F20; 20L15

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Introduction

The study of covering spaces, G-bundles with G a topological group and vector bundles plays an important role in topology and geometry. In this paper, we have introduced the new notion of $(\mathcal{C}, \eta)$-bundle over a space X, where $\mathcal{C}$ is a category and $\eta: \mathcal{C} \to Top$ is a faithful functor into the category Top of spaces.

Particular cases of this notion are the standard notion of covering space (if X is a CW-complex), the notion of overlaying space introduced by Fox [2] (for X a compact metrizable space), the standard construction of suspension foliation and some particular vector bundles that we call flat vector bundles. The category $\mathcal{C}$ plays the role of the structural group of a principal G-bundle.

One advantage of our approach is that our theory and theorems are developed for any general topological space X without any typical conditions of local connectedness.

In this paper, the fundamental progroupoid plays for the classification theorems the role that the fundamental group plays for the standard covering spaces. This fact establishes a natural connection with the shape theory introduced by Borsuk and used by Fox [2] to classify overlaying spaces with a finite numbers of leaves. On the other hand, we use locally constant presheaves in order to have a good relationship with some elementary theory of categories.

The main result of the paper is Theorem 3.6 that gives an equivalence between the category of $(\mathcal{C}, \eta)$-bundles over X and the category pro$(\pi CX, \mathcal{C})$ which is determined by the fundamental progroupoid πCX of X and the category $\mathcal{C}$. Where CX is the prospace associated with X when we consider the Čech nerve $CX(\mathcal{U})$ of all covering reduced sieves directed by refinement.

In foliation theory is very useful to consider coverings manifolds $p: \widetilde{M} \to M$ of a foliated manifold M. In general the saturated subsets S of M have very bad properties of local connectivity. Therefore the restrictions maps $p: p^{-1}S \to S$ are covering spaces that in general can not been classified by the fundamental group of S. Our theory of $(\mathcal{C}, \eta)$-bundles is good enough to deal with these examples which are not included in the standard theory of covering spaces. In order to motivate the sort of coverings we want to classify, consider the following simple example which is a Fox overlaying space with an infinite number of leaves:

Let I be the unit closed interval. In Aut(I) take the map $h(t) = t^2$. In $\mathbb{R} \times I$ consider the discontinuous group generated by

$$H: \mathbb{R} \times I \to \mathbb{R} \times I,$$
$$(x, t) \mapsto (x - 1, t^2).$$

This action induces a natural projection

$$p: \mathbb{R} \times I \to S^1 \times I$$

that is illustrated in Fig. 1.

The foliation induced on $S^1 \times I$ is said to be the suspension of the representation $\eta: \pi_1(S^1 \times I) \to \mathrm{Aut}(F)$ defined for x generator of $\pi_1(S^1 \times I)$ as $\eta(x) = h$. If we

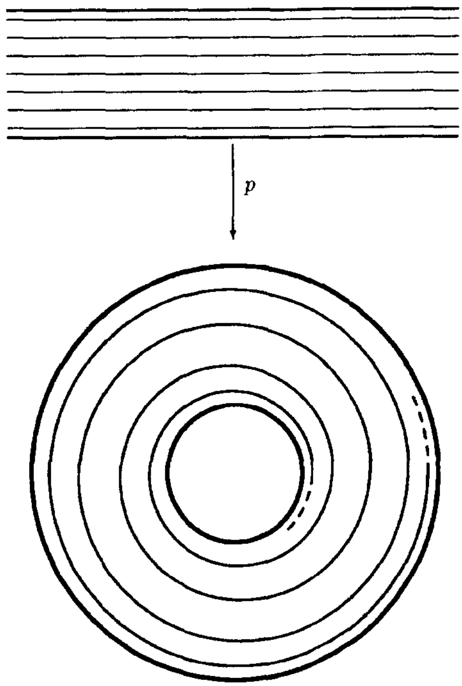


Fig. 1. Overlaying with an infinite number of leaves.

consider $X = \mathrm{Cl}(L) = (S^1 \times \{0\}) \cup L \cup (S^1 \times \{1\})$ where L is a noncompact leaf of $S^1 \times I$, then $p: p^{-1}X \to X$ is a sort of covering space of X, which is not a locally connected space. Fox [2] gives a classification theorem for this kind of overlaying spaces with a finite number of leaves, but in this example p has a numerable infinite number of leaves.

In this paper we not only generalize Fox's classification theorem of overlays in order to include an arbitrary number of leaves, but also develop a theory general enough to deal with many other frequent and interesting topological structures. In particular, for this example in Subsection 4.2.1, we see that the isomorphism classes of overlaying spaces of X with a countable number of leaves are given by the set of conjugation classes of representations of the form $\mathbb{Z} \star \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z})$.

1. Preliminaries

In this section, we introduce some notation and terminology which is frequently used in this paper.

Given a family $\mathcal{U}$ of open subsets of the space X, it is said that $\mathcal{U}$ is a sieve on X if $U \in \mathcal{U}$ and V is an open subset such that $V \subset U$, then $V \in \mathcal{U}$. If $\mathcal{U}$ satisfies that $X = \bigcup_{U \in \mathcal{U}} U$, it is said that $\mathcal{U}$ is a covering sieve on X. We denote by $\mathcal{O}$ the covering sieve of all open subsets of X.

Let $\mathcal{U}$ be a family of nonempty open subsets of X such that if $U \in \mathcal{U}$ and $\emptyset \neq V \in \mathcal{O}$, $V \subset U$, then $V \in \mathcal{U}$. We say that $\mathcal{U}$ is a reduced sieve on X and if $X = \bigcup_{U \in \mathcal{U}} U$, we say that $\mathcal{U}$ is a covering reduced sieve on X.

We note that if $\mathcal{U}$ is a covering sieve on X, then ${}^*\mathcal{U} = \mathcal{U} \setminus \{\emptyset\}$ is a covering reduced sieve on X. An open covering ϑ of X always generates a covering sieve

$$s\vartheta = \{U \in \mathcal{O} \mid \text{there is } V \in \vartheta \text{ such that } U \subset V\}$$

and the corresponding covering reduced sieve ${}^*s\vartheta = \{U \in \mathcal{O} \mid U \neq \emptyset$ and there is $V \in \vartheta$ such that $U \subset V\}$.

A (reduced) sieve $\mathcal{U}$ can be considered as small category denoted again by $\mathcal{U}$, where set of morphisms from U to V is given by $\mathrm{Hom}_{\mathcal{U}}(U,V) = 1$ if $U \subset V$, and otherwise $\mathrm{Hom}_{\mathcal{U}}(U,V) = \emptyset$.

Let $\mathcal{U}$ be a covering reduced sieve on X, and $\mathcal{C}$ a small category. As usual, we denote by $\mathcal{C}^{\mathcal{U}^{op}}$ the category whose objects are all functors $P : \mathcal{U}^{op} \to \mathcal{C}$ and morphisms $P \to P'$ all the natural transformations $\theta : P \to P'$ between such functors. A functor $P : \mathcal{U}^{op} \to \mathcal{C}$ is also called a presheaf on $\mathcal{U}$. A presheaf $P : \mathcal{U}^{op} \to \mathcal{C}$ is said to be locally constant if P carries any arrow $U \subset V$ in $\mathcal{U}$ into an isomorphism $P_U^V : P(V) \to P(U)$. We denote by $(\mathcal{C}^{\mathcal{U}^{op}})_{lc}$ the category of locally constant presheaves on $\mathcal{U}$ (subcategory of $\mathcal{C}^{\mathcal{U}^{op}}$). For more details on presheaves and locally constant presheaves we refer to [6].

We shall also need some notions and properties of procategories and shape theory. For these subjects we refer the reader to [7] and [1]. In particular, we recall that for a given category $\mathcal{C}$ an object of $\mathrm{pro}\,\mathcal{C}$ which is called a proobject is a functor $X : I \to \mathcal{C}$, where I is a small left filtering category. A morphism from $X : I \to \mathcal{C}$ to $X' : I' \to \mathcal{C}$ can be represented by a pair $(\varphi, f(i'))$, where $\varphi : I' \to I$ is map and $f(i')$ denotes a family of maps $f(i') : X(\varphi(i')) \to X(i')$, $i' \in I'$, such that if $i' \to j'$ is an arrow in I', then there are $k \in I$ and arrows $k \to \varphi(i')$, $k \to \varphi(j')$ such that the composite $X(k) \to X(\varphi(i')) \to X(i') \to X(j')$ is equal to the composite $X(k) \to X(\varphi(j')) \to X(j')$. The full subcategory determined by proobjects whose indexing category is $\mathbb{N} = \{\cdots \to 3 \to 2 \to 1 \to 0\}$ will be denoted by $\mathrm{tow}\mathcal{C}$ and its objects will be called towers.

2. The category $\mathrm{pro}(G, \mathcal{C})$

In this section, for a given category $\mathcal{C}$ we define and study the category $\mathrm{pro}(G, \mathcal{C})$, where G is a progroupoid. Later in Section 3 we shall prove that the category of $(\mathcal{C}, \eta)$-bundles is equivalent to a category of the form $\mathrm{pro}(G, \mathcal{C})$.

Recall that a groupoid G is a small category where any morphism in G is an isomorphism. Given two groupoids G, G' a groupoid homomorphism is just a functor $f : G \to G'$. Let Gpd denote the category of groupoids.

We denote by $[0,1]$ the groupoid with two objects 0, 1 and whose morphisms are the identities and two inverse maps $u : 0 \to 1$, $u^{-1} : 1 \to 0$. If G is a groupoid, we can consider the product groupoid $G \times [0,1]$ and the groupoid homomorphisms $\partial_0, \partial_1 : G \to G \times [0,1]$, where, for example, ∂_0 carries an arrow $\alpha : U \to U'$ in G to the arrow

$\partial_0\alpha = (\alpha, \mathrm{id}_0) : (U, 0) \to (U', 0)$. Using this cylinder, we can consider homotopies taking commutative diagrams of the form

$$\begin{array}{ccc} G + G & \xrightarrow{f+g} & G' \\ {\scriptstyle \partial_0+\partial_1}\downarrow & \nearrow{\scriptstyle F} & \\ G \times [0,1] & & \end{array} ,$$

where $G + G$ is the sum groupoid, and F is a groupoid homomorphism.

We note that a homotopy F determines a natural transformation η_F from f to g by $\eta_F(U) = F(\mathrm{id}_U, u)$. Conversely, a natural transformation $\eta : f \to g$ determines a homotopy F_η from f to g by $F_\eta(\mathrm{id}_U, u) = \eta(U)$.

If G, G' are two groupoids, we have the groupoid $\mathrm{HOM}_{Gpd}(G, G')$ whose objects are given by the elements of the set $\mathrm{Hom}_{Gpd}(G, G')$ and if $f, g : G \to G'$ are objects in $\mathrm{HOM}_{Gpd}(G, G')$ a morphism $\eta : f \to g$ is a natural transformation from f to g. We denote by $\pi_0 \,\mathrm{HOM}_{Gpd}(G, G')$ the set of isomorphism classes of the groupoid $\mathrm{HOM}_{Gpd}(G, G')$. The set $\pi_0 \,\mathrm{HOM}_{Gpd}(G, G')$ is also the set of homotopy classes of groupoid homomorphisms from G to G'. We also consider the category $\pi_0 Gpd$ which has the same objects than Gpd and the hom-set is defined by

$$\mathrm{Hom}_{\pi_0 Gpd}(G, G') = \pi_0 \;\mathrm{HOM}_{Gpd}(G, G').$$

Denote by $\gamma : Gpd \to \pi_0 Gpd$ the projection functor which carries an arrow $f : G \to G'$ to the homotopy class $\gamma f : \gamma G \to \gamma G'$. We note that f is an equivalence (of categories) if and only if f is a homotopy equivalence; that is, if γf is an isomorphism in $\pi_0 Gpd$.

For a given progroupoid $G : I \to Gpd$, we consider the category $(G, \mathcal{C})$. An object of $(G, \mathcal{C})$ is given by a pair $(G(i), F)$ where i is an object in I and $F : G(i) \to \mathcal{C}$ is a functor. A morphism α from $(G(i), F)$ to $(G(j), H)$ is a pair $\alpha = (i \to j,\ \theta_\alpha : F \to H\, G_j^i)$ where $i \to j$ is a morphism in I and $\theta_\alpha : F \to H\, G_j^i$ is a natural transformation (G_j^i is the corresponding bonding map).

Consider the class

$$\Sigma = \{\alpha \mid \alpha \text{ is a morphism in } (G, \mathcal{C}) \text{ and } \theta_\alpha \text{ is an equivalence}\}.$$

It is easy to check that the class Σ admits a calculus of right fractions, see [5]. Therefore we can consider the category of right fractions $(G, \mathcal{C})\Sigma^{-1}$ that will be denoted by $\mathrm{pro}(G, \mathcal{C})$.

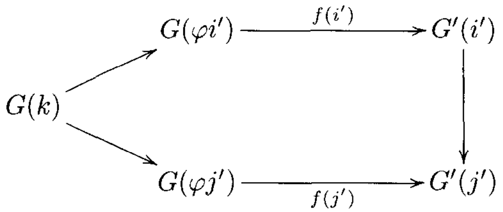
If I is the indexing category of the progroupoid G, and i, j are two objects in I, we consider the category $I \downarrow \{i, j\}$ whose objects are given by a pair of maps (u, v), $u : k \to i$, $v : k \to j$ and a morphism from (u, v) to (u^1, v^1) is given by a map $w : k \to k^1$ such that $u^1 w = u$, $v^1 w = v$. If $(G(i), F)$, $(G(j), H)$ are two objects in $\mathrm{pro}(G, \mathcal{C})$, we take the category $I \downarrow \{i, j\}$, for an object (u, v) in $I \downarrow \{i, j\}$, and we write $k = \mathrm{domain}(u) = \mathrm{domain}(v)$. From the definition of the hom-set in a category of right fractions, one has that:

$$\mathrm{Hom}_{\mathrm{pro}(G,\mathcal{C})}\big((G(i), F), (G(j), H)\big) \cong \operatorname*{colim}_{I\downarrow\{i,j\}} \mathrm{Hom}_{\mathcal{C}^{G(k)}}\big(F G_i^k, H G_j^k\big).$$

Now assume that $f : G \to G'$ is a morphism in pro Gpd represented by a pair $(\varphi, f(i'))$. We are going to see how the pair $(\varphi, f(i'))$ induces a functor $(\varphi, f(i'))^* : \mathrm{pro}(G', \mathcal{C}) \to \mathrm{pro}(G, \mathcal{C})$. First we define a functor from $(G', \mathcal{C})$ to $\mathrm{pro}(G, \mathcal{C})$. Let

$$\alpha' = (i' \to j', \theta_{\alpha'} : F' \to H' G'^{i'}_{j'})$$

be a morphism in $(G', \mathcal{C})$ from $(G'(i'), F')$ to $(G'(j'), H')$, then $(\varphi, f(i'))^*$ carries these objects to $(G(\varphi i'), F' f(i'))$, $(G(\varphi j'), H' f(j'))$, respectively. In order to give $(\varphi, f(i'))^*(\alpha')$ we choose k in I and arrows $k \to \varphi i'$, $k \to \varphi j'$ such that the diagram



is commutative. Then $(\varphi, f(i'))^*(\alpha')$ is the morphism of $\mathrm{pro}(G, \mathcal{C})$ represented by the natural transformation

$$\theta_{\alpha'} * \left(f(i') G^k_{\varphi i'}\right) : F' f(i') G^k_{\varphi i'} \to H' f(j') G^k_{\varphi j'}.$$

It is easy to check that two choices of k represent the same morphism in $\mathrm{pro}(G, \mathcal{C})$. The functor $(\varphi, f(i'))^*$ satisfies that if α' is in Σ', then $(\varphi, f(i'))^*(\alpha')$ is an isomorphism. Therefore we have an induced functor

$$(\varphi, f(i'))^* : \mathrm{pro}(G', \mathcal{C}) \to \mathrm{pro}(G, \mathcal{C}).$$

We note that if $(\varphi, f(i'))$ and $(\psi, g(i'))$ represent the same morphism $f : G \to G'$, then the functor $(\varphi, f(i'))^*$ is isomorphic to the functor $(\psi, g(i'))^*$. We will denote by $f^* : \mathrm{pro}(G', \mathcal{C}) \to \mathrm{pro}(G, \mathcal{C})$ one of these functors.

If $f : G \to G'$ and $g : G' \to G''$ are morphisms in pro Gpd represented by pairs $(\varphi, f(i'))$, $(\psi, g(i''))$, then gf can be represented by $(\varphi\psi, g(i'') f(\psi i''))$. In the case that $gf = \mathrm{id}$ and $fg = \mathrm{id}$, we have that $(\varphi\psi, g(i) f(\psi i))^*$, $(\psi\varphi, f(i') g(\varphi i'))^*$ are isomorphic to identity functors. Therefore the functor $(\varphi, f(i))^*$ is an equivalence of categories.

We give some properties of the category $\mathrm{pro}(G, \mathcal{C})$ that will be used in this paper.

Lemma 2.1. *If $f : G \to G'$ is an isomorphism in* pro *Gpd, then $f^* : \mathrm{pro}(G', \mathcal{C}) \to \mathrm{pro}(G, \mathcal{C})$ is an equivalence of categories.*

Lemma 2.2. *Let $f : G \to G'$ be a level morphism in* pro *Gpd such that for each $i \in I$, $f(i) : G(i) \to G'(i)$ is an equivalence, then $f^* : \mathrm{pro}(G', \mathcal{C}) \to \mathrm{pro}(G, \mathcal{C})$ is an equivalence of categories.*

Proposition 2.3. *Let G, G' be objects in* tow *$\pi_0 Gpd$. If G is isomorphic to G' in* tow *$\pi_0 Gpd$, then $\mathrm{pro}(G, \mathcal{C})$ is equivalent to $\mathrm{pro}(G', \mathcal{C})$.*

Remark. These results can be obtained changing in [4] the category *Sets* by the category $\mathcal{C}$.

3. $(\mathcal{C}, \eta)$-bundles

In this section we define a notion of $(\mathcal{C}, \eta)$-bundle that generalizes the one of covering projections in [4].

We will consider the category *Top* of topological spaces and continuous maps, a general category $\mathcal{C}$, and a faithful functor $\eta : \mathcal{C} \to Top$.

Let $p : E \to X$ a morphism in *Top* and $\mathcal{U}$ a covering reduced sieve of X. A $(\mathcal{C}, \eta)$-atlas $\mathcal{A}$ for $p : E \to X$ on $\mathcal{U}$ is a pair $(P, \{\varphi_U\}_{U \in \mathcal{U}})$ where $P : \mathcal{U}^{op} \to \mathcal{C}$ is a locally constant presheaf and $\{\varphi_U : p^{-1}U \to U \times \eta P(U) \mid U \in \mathcal{U}\}$ is a family of isomorphisms over U such that if $U \subset V$, $U, V \in \mathcal{U}^{op}$ the following diagram is commutative:

$$\begin{array}{ccc} & p^{-1}U & \\ {}^{\varphi_U}\swarrow & & \searrow^{\varphi_V|_{p^{-1}U}} \\ U \times \eta P(U) & \xleftarrow[\mathrm{id}_U \times \eta P_U^V]{} & U \times \eta P(V) \end{array}.$$

Two atlases $\mathcal{A} = (P, \{\varphi_U\}_{U \in \mathcal{U}})$ and $\mathcal{B} = (Q, \{\psi_V\}_{V \in \mathcal{V}})$ are said to be equivalent if there exists a covering reduced sieve $\mathcal{W}$ refining both $\mathcal{U}$ and $\mathcal{V}$, and a natural equivalence $\theta : P|_{\mathcal{W}^{op}} \to Q|_{\mathcal{W}^{op}}$ such that the following diagram is commutative:

$$\begin{array}{ccc} & p^{-1}W & \\ {}^{\varphi_W}\swarrow & & \searrow^{\psi_W} \\ W \times \eta P(W) & \xleftarrow[\mathrm{id}_W \times \eta\theta(W)]{} & W \times \eta Q(W) \end{array}.$$

Definition 3.1. A $(\mathcal{C}, \eta)$-bundle over X is a pair $(p : E \to X, [\mathcal{A}])$ with p morphism in *Top* and $[\mathcal{A}]$ an equivalence class of atlases.

Definition 3.2. Let $\mathcal{P} = (p : E \to X, [\mathcal{A}])$, $\mathcal{P}' = (p' : E \to X, [\mathcal{A}'])$ be two $(\mathcal{C}, \eta)$-bundles. A $(\mathcal{C}, \eta)$-bundle transformation $f : \mathcal{P} \to \mathcal{P}'$ is a continuous map $f : E \to E'$ such that $p'f = p$ and there exist a covering reduced sieve $\mathcal{W}$ of X, atlases $\mathcal{A}$ for p and $\mathcal{A}'$ for p' on $\mathcal{W}$ and a natural transformation $\theta : P \to P'$ such that $\forall W \in \mathcal{W}$ diagram below is commutative

$$\begin{array}{ccc} p^{-1}W & \xrightarrow{\quad f \quad} & (p')^{-1}W \\ {\scriptstyle \varphi_W}\downarrow & & \downarrow{\scriptstyle \varphi'_W} \\ W \times \eta P(W) & \xrightarrow[\mathrm{id}_W \times \eta\theta(W)]{} & W \times \eta P'(W) \end{array}.$$

We shall denote by $(\mathcal{C},\eta)$-bundle(X) the category of $(\mathcal{C},\eta)$-bundles and bundle transformations. If $\mathcal{P}$ is a $(\mathcal{C},\eta)$-bundle we will call $\mathcal{C}$ the structural category of $\mathcal{P}$.

Let $\mathcal{U}$ be a covering reduced sieve on X. A $(\mathcal{C},\eta)$-bundle $\mathcal{P}$ is said trivial on $\mathcal{U}$ if $\mathcal{P}$ admits an atlas on $\mathcal{U}$. We also say that $\mathcal{P}$ trivializes on $\mathcal{U}$. A morphism $f:\mathcal{P}\to\mathcal{P}'$ is said to be trivial on $\mathcal{U}$ if there exists atlases $\mathcal{A}$ of $\mathcal{P}$ and $\mathcal{A}'$ of $\mathcal{P}'$ on $\mathcal{U}$ and a natural transformation $\theta:P\to P'$ such that for any $W\in\mathcal{U}$ diagram above is commutative. We denote by $((\mathcal{C},\eta)\text{-bundle}(X))_{\mathcal{U}}$ the subcategory of $(\mathcal{C},\eta)$-bundle(X) whose objects and maps trivialize on $\mathcal{U}$.

Theorem 3.3. *$(\mathcal{C}^{\mathcal{U}^{op}})_{lc}$ is equivalent to $((\mathcal{C},\eta)\text{-bundle}(X))_{\mathcal{U}}$.*

Proof. Let P be a locally constant presheaf on $\mathcal{U}$. We define the functor

$$\mathcal{E}:(\mathcal{C}^{\mathcal{U}^{op}})_{lc}\to\big((\mathcal{C},\eta)\text{-bundle}(X)\big)_{\mathcal{U}}\quad\text{by}$$
$$\mathcal{E}(P)=\big(p(P):E(P)\to X,[\mathcal{A}(P)]\big),$$

where $E(P)$, $p(P)$ and $\mathcal{A}(P)$ are defined as follows:

In $\bigsqcup_{U\in\mathcal{U}}(U\times\eta P(U))$ we consider the following relation. Given $(u,x)\in(U\times\eta P(U))$, $(v,y)\in(V\times\eta P(V))$, $(u,x)\sim(v,y)$ if $u=v$ and exists $W\subset U\cap V$ with $u\in W$ and $\eta P^U_W(x)=\eta P^V_W(y)$. It is easy to check that this is an equivalence relation, so we can define

$$E(P)=\bigsqcup_{U\in\mathcal{U}}(U\times\eta P(U))/\sim\quad\text{and}\quad p(P)[(u,x)]=u.$$

In order to obtain the atlas, we will consider the presheaf $P:\mathcal{U}^{op}\to\mathcal{C}$. We have

$$\pi:\bigsqcup_{U\in\mathcal{U}}(U\times\eta P(U))\to E(P).$$

For $U\in\mathcal{U}$ we will prove that

$$\pi_U=\pi|_{(U\times\eta P(U))}:\big(U\times\eta P(U)\big)\to p(P)^{-1}(U)$$

is an isomorphism in *Top* therefore the charts $\varphi(P)_U$ will be defined by $\varphi(P)_U=\pi_U^{-1}$.

π_U is injective: Let $[(u,x)]=[(u,y)]$ with $(u,x),(v,y)\in U\times\eta P(U)$ then $u=v$ and there exists $W\subset U$ such that $\eta P^U_W(x)=\eta P^U_W(y)$ since η is faithful and P^U_W is an isomorphism it follows that $x=y$.

π_U is surjective: Let $[(u,x)]\in p^{-1}(P)U$ then $p(P)[(u,x)]=u\in U$ we have that $(u,x)\in V\times\eta PV$ for some $V\in\mathcal{U}$. Let be $W=U\cap V$ then $(u,x)\sim(u,\eta P^U_W(x))\sim(u,\eta(P^U_W)^{-1}(P^V_W)(x))\in(U\times\eta PU)$.

π_U is continuous: This follows from the continuity of π.

π_U is an open map: Let $(V\times W)\in U\times\eta P(U)$ a basic open set. $\pi_U(V\times W)$ is open in $E(P)$ if $\pi_V^{-1}(\pi_U(V\times W))$ is open in $\bigsqcup_{U\in\mathcal{U}}(U\times\eta P(U))$ but $\pi_V^{-1}(\pi_U(V\times W))=(U'\cap V)\times(P^{U'}_{U'\cap V})^{-1}P^U_{U'\cap V}W$ is an open set for any $U'\in\mathcal{U}$. Therefore π_U is an homeomorphism.

Now we prove that $\mathcal{E}:(\mathcal{C}^{\mathcal{U}^{op}})_{lc}\to((\mathcal{C},\eta)\text{-bundle}(X))_{\mathcal{U}}$ is a full embedding. Let $P,Q\in(\mathcal{C}^{\mathcal{U}^{op}})_{lc}$.

$\mathcal{E}$ *is faithful*: If $\theta, \mu : P \to Q$ are natural transformations for each $U \in \mathcal{U}$ we have:

$$\begin{array}{ccccc} U \times \eta P(U) & \xleftarrow{\varphi_U} & p(P)^{-1}U & \xrightarrow{\varphi_U} & U \times \eta P(U) \\ \downarrow{\scriptstyle \mathrm{id} \times \eta\theta(U)} & & \downarrow{\scriptstyle \mathcal{E}(\theta)} \downarrow{\scriptstyle \mathcal{E}(\mu)} & & \downarrow{\scriptstyle \mathrm{id} \times \eta\mu(U)} \\ U \times \eta Q(U) & \xleftarrow[\psi_U]{} & p(Q)^{-1}U & \xrightarrow[\psi_U]{} & U \times \eta Q(U) \end{array}$$

If $\mathcal{E}(\theta) = \mathcal{E}(\mu)$ then $\eta\theta(U) = \eta\mu(U)$ and now by fidelity of η, $\theta(U) = \mu(U)$.

$\mathcal{E}$ *is full*: Just remind that a morphism $f : \mathcal{E}(P) \to \mathcal{E}(Q)$ is a continuous map $f : E(P) \to E(Q)$ such that $p(Q)f = p(P)$ together with atlases $\mathcal{A}$, $\mathcal{B}$ over $\mathcal{U}$ and a natural transformation $\theta : P \to Q$ which for any $U \in \mathcal{U}$ the associated diagram (see Definition 3.1) is commutative. It is easy to check that θ is such that $\mathcal{E}(\theta) = f$ so $\mathcal{E}$ is full.

$\mathcal{E}$ *is essentially surjective*: Given $\mathcal{P} \in ((\mathcal{C}, \eta)\text{-bundle}(X))_{\mathcal{U}}$ with atlas

$$\mathcal{A} = (Q, \{\varphi_U\}_{U \in \mathcal{U}})$$

we will see that $\mathcal{E}(Q) \cong \mathcal{P}$. The family of homeomorphisms $\{\varphi_U^{-1} : U \times \eta Q(U) \to p^{-1}U \mid U \in \mathcal{U}\}$ induce a map

$$f = \bigsqcup_{U \in \mathcal{U}} f|_{U \times \eta Q(U)} : \bigsqcup_{U \in \mathcal{U}} (U \times \eta Q(U)) \to E$$

with $f|_{U \times \eta Q(U)} = \varphi_U^{-1}$. Now we can consider $\{\bar{f}_U = \varphi_U^{-1}\varphi(P)_U : p(Q)^{-1}U \to p^{-1}U \mid U \in \mathcal{U}\}$. This family of homeomorphisms defines a continuous map $\bar{f} : E(Q) \to E$ such that $\bar{f}\pi = f$ and we have that

$$\begin{array}{ccc} p(Q)^{-1}U & \xrightarrow{\bar{f}} & (p)^{-1}U \\ \downarrow{\scriptstyle \varphi(P)_U} & & \downarrow{\scriptstyle \varphi_U} \\ U \times \eta Q(U) & \xrightarrow[\mathrm{id} \times \mathrm{id}]{} & U \times \eta Q(U) \end{array}$$

commutes, so $\bar{f}$ is an isomorphism; i.e., $\mathcal{P}$ is isomorphic to $\mathcal{E}(Q)$. Therefore one has that $\mathcal{E}$ is essentially surjective which completes the proof of the theorem. □

Given a covering reduced sieve $\mathcal{U}$, if we take the class Σ of all morphisms in $\mathcal{U}$, we have the corresponding category of fractions $\pi\mathcal{U} = \mathcal{U}[\Sigma^{-1}]$ which is a groupoid. We note the existence of natural isomorphisms $(\pi\mathcal{U})^{op} \cong \pi(\mathcal{U}^{op})$. Thus we use the notation $\pi\mathcal{U}^{op}$. For locally constant presheaves on $\mathcal{U}$, one has:

Lemma 3.4. *Given a covering reduced sieve $\mathcal{U}$ on X, the category $(\mathcal{C}^{\mathcal{U}^{op}})_{lc}$ of locally constant presheaves on $\mathcal{U}$ is equivalent to the functor category $\mathcal{C}^{\pi\mathcal{U}^{op}}$.*

Proof. Denote by $\gamma : \mathcal{U}^{op} \to \pi\mathcal{U}^{op}$ the projection functor. If $P : \mathcal{U}^{op} \to \mathcal{C}$ is locally constant, then P carries any arrow of $\mathcal{U}^{op}$ into an isomorphism. Therefore, P factors through $\pi\mathcal{U}^{op}$ as $P = \overline{P}\gamma$. Conversely, if $F : \pi\mathcal{U}^{op} \to \mathcal{C}$ is a functor, because $\pi\mathcal{U}^{op}$ is a groupoid F carries any arrow of $\pi\mathcal{U}^{op}$ into an isomorphism. Then $F\gamma$ is a locally constant functor. □

If $\mathcal{U}$, ϑ are open coverings of a space X. It is said that $\mathcal{U}$ refines ϑ, $\mathcal{U} \geqslant \vartheta$ if for any $U \in \mathcal{U}$, there is $V \in \vartheta$ such that $U \subset V$. We note that for a given U, in general it is possible to find various $V \in \vartheta$ such that $U \subset V$. It would be interesting to have a canonical way of finding a V for each U. We solve this problem if we work only with covering reduced sieves. We note that if $\mathcal{U}$, ϑ are two covering reduced sieves then $\mathcal{U}$ refines ϑ if and only if $\mathcal{U} \subset \vartheta$. If $U \in \mathcal{U}$, there is $V \in \vartheta$ such that $U \subset V$, but this implies that $U \in \vartheta$. If $\mathcal{U} \subset \vartheta$, there is an induced functor $\pi_{\vartheta}^{\mathcal{U}} : \pi\mathcal{U}^{op} \to \pi\vartheta^{op}$ that again induces a functor $\mathcal{C}^{\pi\mathcal{U}^{op}} \to \mathcal{C}^{\pi\mathcal{V}^{op}}$.

Using the equivalence of categories $\mathcal{C}^{\pi\mathcal{U}^{op}} \xrightarrow{\gamma^*} (\mathcal{C}^{\mathcal{U}^{op}})_{lc}$ we have a new equivalence $\mathcal{E}'$ obtained as the composite $\mathcal{E}' = \mathcal{E}\gamma^*$

$$\mathcal{C}^{\pi\mathcal{U}^{op}} \xrightarrow{\gamma^*} \left(\mathcal{C}^{\mathcal{U}^{op}}\right)_{lc} \xrightarrow{\mathcal{E}} \left((\mathcal{C}, \eta)\text{-bundle}(X)\right)_{\mathcal{U}}.$$

If $\mathcal{U}$ refines ϑ one has the following:

Proposition 3.5. *Let $\mathcal{U}$, ϑ be two covering reduces sieves on X. If $\mathcal{U} \subset \vartheta$, then the functor diagram*

$$\begin{array}{ccc} \mathcal{C}^{\pi\vartheta^{op}} & \xrightarrow{\mathcal{E}'} & ((\mathcal{C}, \eta)\text{-bundle}(X))_{\vartheta} \\ \downarrow & & \downarrow \\ \mathcal{C}^{\pi\mathcal{U}^{op}} & \xrightarrow[\mathcal{E}']{} & ((\mathcal{C}, \eta)\text{-bundle}(X))_{\mathcal{U}} \end{array}$$

is commutative up to natural isomorphism.

Proof. If F is an object in $\mathcal{C}^{\pi\vartheta^{op}}$, we have the presheaf $P = \gamma F$, that satisfies that

$$\operatorname*{colim}_{x \in V \in \vartheta} P(V) \cong \operatorname*{colim}_{x \in U \in \mathcal{U}} P(U).$$

From this fact, it follows easily the existence of a isomorphism of functors in the diagram above. □

Given a space X, we denote by $COV(X)$ the set of open coverings $\mathcal{U}$ of X directed by refinement. We denote by $CRS(X)$ the set of covering reduced sieves of X directed by refinement or equivalently by "inclusion"; that is if $\mathcal{U}, \vartheta \in CRS(X)$ then $\mathcal{U} \geqslant \vartheta$ if and only if $\mathcal{U} \subset \vartheta$. Recall that *Gpd* denotes the category of groupoids. Using the directed set $CRS(X)$ as an indexing category we can consider the progroupoid

$$\pi\, crs(X) : CRS(X) \to Gpd$$

defined by $\pi\, crs(X)(\mathcal{U}) = \pi\mathcal{U}^{op}$. Associated with the progroupoid $\pi\, crs(X)$ we have the category $\text{pro}(\pi\, crs(X), \mathcal{C})$ defined in Section 1. The main result, of this section is the following:

Theorem 3.6. *The category* $(\mathcal{C}, \eta)$-bundle(X) *is equivalent to the category*

$$\text{pro}(\pi\, crs(X), \mathcal{C}).$$

Proof. For the groupoid $\pi\, crs(X)$, consider the category $(\pi\, crs(X), \mathcal{C})$ defined in Section 1. Now we are going to define a functor

$$\mathcal{E}' : \big(\pi\, crs(X), \mathcal{C}\big) \to (\mathcal{C}, \eta)\text{-bundle}(X).$$

Suppose that $(\pi\mathcal{U}^{op}, F)$, $(\pi\vartheta^{op}, G)$ are objects in $(\pi\, crs(X), \mathcal{C})$ and $\alpha = (\mathcal{U} \subset \vartheta, \theta_\alpha : F \to G\,\pi_\vartheta^{\mathcal{U}})$ is a morphism in $(\pi\, crs(X), \mathcal{C})$. The functor $\mathcal{E}'$ carries

$$\alpha : (\pi\,\mathcal{U}^{op}. F) \to (\pi\,\vartheta^{op}, G)$$

to $\mathcal{E}'\alpha : \mathcal{E}'(\pi\,\mathcal{U}^{op}, F) \to \mathcal{E}'(\pi\,\vartheta^{op}, G)$, where $\mathcal{E}'(\pi\,\mathcal{U}^{op}, F) = \mathcal{E}(F\gamma)$, $\mathcal{E}'(\pi\,\vartheta^{op}, G) = \mathcal{E}(G\gamma)$ and if $s \in F\gamma(U)$, $x \in U \in \mathcal{U}$,

$$\mathcal{E}'\alpha\big(\pi_U(u, x)\, s\big) = \pi_U(u, x)\big(\theta_\alpha * \gamma(U)(s)\big).$$

It is easy to check that if α is in Σ, then $\mathcal{E}'\alpha$ is an isomorphism, hence there is an induced functor

$$\mathcal{E}' : \text{pro}\big(\pi\, crs(X), \mathcal{C}\big) \to (\mathcal{C}, \eta)\text{-bundle}(X).$$

We note that if $\mathcal{U}$, ϑ are in $CRS(X)$, then $\mathcal{U} \cap \vartheta = \{W \mid W \in \mathcal{U} \text{ and } W \in \vartheta\}$ is in $CRS(X)$. We also have that the inclusion of directed sets $CRS(X) \to COV(X)$ is cofinal.

If $(\pi\,\mathcal{U}^{op}, F)$, $(\pi\,\vartheta^{op}, G)$ are objects in $\text{pro}(\pi\, crs(X), \mathcal{C})$, one has that

$$\begin{aligned}
&\text{Hom}_{\text{pro}(\pi\, crs(X), \mathcal{C})}\big((\pi\,\mathcal{U}^{op}, F), (\pi\,\vartheta^{op}, G)\big) \\
&\quad\cong \operatorname*{colim}_{\mathcal{W} \in CRS(X),\, \mathcal{W} \subset \mathcal{U} \cap \vartheta} \text{Hom}_{\mathcal{C}^{\pi\mathcal{W}^{op}}}\big(F\,\pi_{\mathcal{U}}^{\mathcal{W}}, G\,\pi_{\vartheta}^{\mathcal{W}}\big) \\
&\quad\cong \operatorname*{colim}_{\mathcal{W} \in CRS(X),\, \mathcal{W} \subset \mathcal{U} \cap \vartheta} \text{Hom}_{((\mathcal{C},\eta)\text{-bundle}(X))_{\mathcal{W}}}\big(\mathcal{E}'(F\,\pi_{\mathcal{U}}^{\mathcal{W}}), \mathcal{E}'(G\,\pi_{\vartheta}^{\mathcal{W}})\big) \\
&\quad\cong \operatorname*{colim}_{\mathcal{W} \in CRS(X),\, \mathcal{W} \subset \mathcal{U} \cap \vartheta} \text{Hom}_{((\mathcal{C},\eta)\text{-bundle}(X))_{\mathcal{W}}}\big(\mathcal{E}'F, \mathcal{E}'G\big) \\
&\quad\cong \text{Hom}_{(\mathcal{C},\eta)\text{-bundle}(X)}\big(\mathcal{E}'F, \mathcal{E}'G\big).
\end{aligned}$$

Thus we have showed that $\mathcal{E}' : \text{pro}(\pi\, crs(X), \mathcal{C}) \to (\mathcal{C}, \eta)\text{-bundle}(X)$ is a full faithful functor.

On the other hand, if $\Phi = (p : E \to X, [\mathcal{A}])$ is an object in $(\mathcal{C}, \eta)$-bundle(X) and $\mathcal{A}$ is an atlas on a covering reduced sieve $\mathcal{U}$. By Theorem 3.3 and Lemma 3.4, there is $F : \pi\,\mathcal{U}^{op} \to C$ such that $\mathcal{E}'(\pi\,\mathcal{U}^{op}, F) \cong \Phi$. Therefore $\mathcal{E}' : \text{pro}(\pi\, crs(X), \mathcal{C}) \to (\mathcal{C}, \eta)$-bundle$(X)$ is an equivalence of categories. $\square$

Let $\mathcal{P}$ be a $(\mathcal{C},\eta)$-bundle. If $\mathcal{C}'$ is a subcategory of $\mathcal{C}$, we will say that $\mathcal{P}$ reduces $\mathcal{C}$ to $\mathcal{C}'$ if there exists an atlas $(P,\{\varphi_U\}_{U\in\mathcal{U}})$ such that the presheaf $P:\mathcal{U}^{op}\to\mathcal{C}$ factorizes through the subcategory $\mathcal{C}'$, i.e., $P=i\overline{P}$ with $\overline{P}:\mathcal{U}^{op}\to\mathcal{C}'$ functor, and $i:\mathcal{C}'\to\mathcal{C}$ the inclusion functor.

Remarks.

(1) The inclusion $i:\mathcal{C}'\to\mathcal{C}$ induces a faithful functor $\text{pro}(\pi,\mathcal{C}')\to\text{pro}(\pi,\mathcal{C})$. The $(\mathcal{C},\eta)$-bundles represented by objects from $\text{pro}(\pi,\mathcal{C})$ isomorphic to objects from the subcategory $\text{pro}(\pi,\mathcal{C}')$ are those which reduce $\mathcal{C}$ to $\mathcal{C}'$.

(2) We note that a $(\mathcal{C},\eta)$-bundle $\mathcal{P}$ reduces $\mathcal{C}$ to $\mathcal{C}'$, the subcategory of identities of $\mathcal{C}$, if and only if $\mathcal{P}$ is trivial; that is, E is a product.

4. Examples and applications

The category $\text{pro}(\pi CX,\mathcal{C})$ depends of the first variable πCX and of the second $\mathcal{C}$. In the Subsection 4.1 we analyze how the progroupoid πCX reduces in some cases to a groupoid, or to a group and in other cases reduces to a progroup or even to a topological group. In the second subsection we analyze the second variable for the cases $\mathcal{C}=$ *Set*, *Top* and the category of $\mathcal{R}$-modules $\mathcal{M}_{\mathcal{R}}$. It is interesting to note that the category $\text{pro}(\pi CX,\mathcal{C})$ does not depend of the embedding functor $\eta:\mathcal{C}\to$ *Top*.

4.1. *The progroupoid* $\pi\,crs(X)$

The Čech nerve $CX(\mathcal{U})$ of an open covering $\mathcal{U}$ of X is defined to be the simplicial set whose q-simplexes are given by:

$$CX(\mathcal{U})_q=\{(U_0,\dots,U_q)\mid U_0,\dots,U_q\in\mathcal{U},\ U_0\cap\dots\cap U_q\neq\emptyset\}.$$

We note that if $\mathcal{U}$, $\mathcal{V}$ are covering reduced sieves and $\mathcal{U}\geqslant\mathcal{V}$, then one has an inclusion $\mathcal{U}\subset\mathcal{V}$ that induces a map $CX(\mathcal{U})\to CX(\mathcal{V})$. This gives a functor $CX:CRS(X)\to SS$ and if $\pi:SS\to Gpd$ is the fundamental groupoid functor, then we have the progroupoid $\pi CX:CRS(X)\to Gpd$. The category $\text{pro}(\pi\,crs(X),\mathcal{C})$ is equivalent to $\text{pro}(\pi CX,\mathcal{C})$ as it is proved in [4]. Therefore in order to classify $(\mathcal{C},\eta)$-bundles, we can use the progroupoid $\pi\,crs(X)$ or the progroupoid πCX.

We note that if X is a locally path-connected and semilocally 1-connected space then $\pi\,crs(X)$ is "weak equivalent" to the standard fundamental groupoid $\pi=\pi(X)$. In this case the category $\text{pro}(\pi\,crs(X),\mathcal{C})$ is equivalent to the functor category $\mathcal{C}^{\pi}$. If X is also connected then $\pi(X)$ is equivalent to the fundamental group $\pi_1=\pi_1(X)$ considered as a category with only one object. Now $\mathcal{C}^{\pi_1}$ is the category of π_1-objects in $\mathcal{C}$.

Given a pointed space (X,x_0), if X is connected, then πCX is "weak equivalent" to the fundamental progroup $\pi_1C(X,x_0)$, obtained by taking pointed open coverings of the form $(\mathcal{U},U_0)$ with $x_0\in U_0\in\mathcal{U}$ and the corresponding pointed refinement.

4.2. Particular cases of $(\mathcal{C}, \eta)$-bundles

Firstly, we take $\mathcal{C}$ the category *Set* of sets and $\eta: \mathit{Set} \to \mathit{Top}$ the functor defined by $\eta(X) = (X, \mathit{dis})$, where *dis* denotes the discrete topology. Secondly, we take $\mathcal{C} = \mathit{Top}$ and $\eta: \mathit{Top} \to \mathit{Top}$, $\eta(X) \to X_{dis}$, X with the discrete topology. Finally we analyze the case that $\mathcal{C}$ is a category of $\mathcal{R}$-modules.

4.2.1. (Set, η)-bundles

In this case we consider the functor $\eta: \mathit{Set} \to \mathit{Top}$ defined by $\eta(X) = (X, \mathit{dis})$. This induces the corresponding notion of (Set, η)-bundle over any topological space X. Nevertheless, if the space X is locally connected this notion is equivalent to the standard one of covering projection given for instance in Spanier's book [9]. Therefore if X is a locally connected space, the category (Set, η)-bundle(X) is the category of covering projections of X. In this case the progroupoid πCX has the additional property of being isomorphic to a progroupoid whose bonding maps are surjections. This kind of surjective progroup have a very good connection with localic groupoids, see [8]. Of course, under good local connectedness condition the progroupoid πCX reduces to the fundamental group. In this case, we have that the standard classification of covering spaces for a connected, locally path-connected and semilocally 1-connected is a particular case of our main theorem. In this case the category of covering spaces is equivalent to the category of π_1-sets where π_1 is the fundamental group of X.

Suppose that X is a compact metrizable space. Then our notion of (Set, η)-bundle is equivalent to the notion of overlay introduced by Fox [2]. Then if X is a compact metrizable space the category (Set, η)-bundle(X) is equivalent to the category of overlays of X. For this case, our classification Theorem 3.6 generalizes Fox's classification theorem. Then the category of "overlays" of X is equivalent to pro$(\pi CX, \mathit{Set})$. With the additional condition of connectedness on X we have that the category of overlays over X is equivalent to pro$(\pi_1 C(X, x_0), \mathit{Set})$ for any base point $x_0 \in X$. For a connected space X, in order to obtain Fox's classification theorem of overlays with n leaves it suffices to consider the full subcategory $\mathit{End}(n)$ of *Set* determined by the finite set $\{1, \ldots, n\}$. In this case, the objects of the category pro$(\pi_1 C(X, x_0), \mathit{End}(n))$ are given by representations of the form $\pi_1 C(X, x_0) \to S_n$, where S_n is the symmetric group of degree n.

Moreover, if (X, x_0) is also a 1-movable pointed space, then pro$(\pi_1 C(X, x_0), \mathit{Set})$ is equivalent to the category of continuous $\check{\pi}_1(X, x_0)$-sets where $\check{\pi}_1(X, x_0) = \lim \pi_1 C(X, x_0)$ is the Čech fundamental group provided with the inverse limit topology.

For example, let X be the space considered in the introduction and take a base point $x_0 \in X$, here we have $\check{\pi}_1(X, x_0)$ is a free group generated by two elements, with the discrete topology. If we want to classify the "overlays" with a numerable number $\aleph_0 = |\mathbb{Z}|$ of leaves, as a consequence of our main theorem we obtain that pro$(\pi_1 C(X, x_0), \mathit{End}(\mathbb{Z})) \simeq \mathrm{Aut}(\mathbb{Z}) \times \mathrm{Aut}(\mathbb{Z})$ considered as a category where the objects are its elements (f, g) and the morphisms are given by maps $u: \mathbb{Z} \to \mathbb{Z}$ such that

$uf = fu$ and $ug = gu$. The overlay of the introduction corresponds to the element (f, f) where $f \in \mathrm{Aut}(\mathbb{Z})$, is given by $f(z) = z + 1$.

4.2.2. (Top, η)-bundles

Let $\eta\colon Top \to Top$ be the functor which carries a space A to $\eta A = A$ with the discrete topology. An important particular case for this functor is the category of suspension foliations with base a manifold X and fiber a manifold F.

In order to have an alternative description of the notion of (Top, η)-bundle we introduction the following concepts:

A topological $(\tau \subseteq \tau')$-space is a set X endowed with two topologies τ, τ' where τ' is finer than τ. It is denoted by $(X, \tau \subseteq \tau')$ and it is also called a $(\tau \subseteq \tau')$-space. A covering $\mathcal{U}$ on X is a family of $U \in \tau$ such that $X = \bigcup_{U \in \mathcal{U}} U$. A $(\tau \subseteq \tau')$-map is a τ-continuous and τ'-continuous map. A $(\tau \subseteq \tau')$-space is said to be discrete if τ' is discrete. A topological space (X, τ) can be considered as a $(\tau \subseteq \tau')$-space taking $\tau = \tau'$.

Given a $(\tau \subseteq \tau')$-map $p\colon E \to X$, where X is a space considered as a $(\tau \subseteq \tau')$-space, an atlas consists on a covering $\mathcal{U}$ and a family of $(\tau \subseteq \tau')$-homeomorphisms $\{\varphi_U : p^{-1}U \to U \times F(U)\}_{U \in \mathcal{U}}$ where each $F(U)$ is a discrete $(\tau \subseteq \tau')$-space.

Two atlases $\mathcal{A} = \{\varphi_U \mid U \in \mathcal{U}\}$ and $\mathcal{B} = \{\psi_V \mid V \in \mathcal{V}\}$ are said equivalent if there exists $\mathcal{W}$ refining both $\mathcal{U}$ and $\mathcal{V}$ such that if $W \subset U \cap V$ then

$$W \times F(U) \xrightarrow{\varphi_U|p^{-1}W} p^{-1}W \xrightarrow{\varphi_V^{-1}} W \times G(U)$$

is W-constant.

A flat bundle over X is a pair $(p\colon E \to X, [\mathcal{A}])$ with p and $\mathcal{A}$ as above. It is easy to check that the notion of (Top, η)-bundle is equivalent to the flat bundle.

If X is a locally connected space, then two atlases of p are always equivalent, so we have that $p\colon E \to X$ is a flat bundle if and only if p trivializes over some $\mathcal{U}$ covering of X with $(\tau \subseteq \tau')$-discrete fiber. We note that we can look at a flat bundle in two ways, we can think as a locally trivial fiber bundle with locally constant cocycle. On the other hand, we can think that a flat bundle is a covering projection with discrete fiber.

If X is locally path-connected and semilocally 1-connected space, the progroupoid $\pi crs(X)$ is equivalent to the standard fundamental groupoid π, and so the category flat-bundle(X) of flat bundles over X is equivalent to the category Top^{π} of π-spaces.

If X is also a connected space, then the fundamental groupoid is equivalent to the fundamental group. Therefore the flat bundle category is equivalent to the π-spaces where π is endowed with the discrete topology. Now we have that flat bundles with fiber F are classified by the representations $\pi \to \mathrm{Aut}(F)$. Suppose that $\mathrm{Aut}(F)$ is provided with the compact open topology and $\mathrm{Aut}(F)_{dis}$ has the discrete topology, then the continuous homomorphism $\mathrm{Aut}(F)_{dis} \to \mathrm{Aut}(F)$ induces on the classifying spaces the map $B(\mathrm{Aut}(F)_{dis}) \to B(\mathrm{Aut}(F))$. Denote by $[\pi, \mathrm{Aut}(F)]$ the conjugation classes of representations of π on $(\mathrm{Aut}(F)_{dis})$, then $[\pi, \mathrm{Aut}(F)_{dis}] \cong [X, B(\mathrm{Aut}(F)_{dis})]$ and the map $B(\mathrm{Aut}(F)_{dis}) \to B(\mathrm{Aut}(F))$ gives for a flat F-bundle its forgetful structure of F-bundle with locally constant cocycle.

We finish this subsection with the following nice example: If X and F are topological manifolds with respective dimensions p, q then the F-flat bundles (with fiber F) over X is the category of suspension foliations with base X and fiber F [3]. In this case, the category $\mathrm{pro}(\pi CX, End(F))$ reduces to the category of representations of the form $\pi \to \mathrm{Aut}(F)$.

4.2.3. $(\mathcal{M}_{\mathcal{R}}, \eta)$*-bundles*

Let $\mathcal{M}_{\mathcal{R}}$ be the category of $\mathcal{R}$-modules. Consider the functor $\eta\colon \mathcal{M}_{\mathcal{R}} \to Top$ defined by $\eta M = (M, dis)$, where dis is the discrete topology. If X is a locally connected space the notion of $(\mathcal{M}_{\mathcal{R}}, \eta)$-bundle is equivalent to the notion of locally constant sheaf of $\mathcal{R}$-modules on X. In particular, we obtain the category of locally constant sheaves of abelian groups on the space X.

Finally consider the cases $\mathcal{R} = \mathbb{R}$, $\mathcal{R} = \mathcal{C}$. Let $V_{\mathcal{R}}$ be the category of finite-dimensional vector spaces over $\mathcal{R}$ and $\eta\colon V_{\mathbb{R}} \to Top$ such that $\eta V = V$ with the standard topology.

A flat vector bundle is a vector bundle whose cocycle is locally constant. It is easy to check that the notion of $(V_{\mathcal{R}}, \eta)$-bundle is equivalent to the notion of flat vector bundle. As another particular case of our classification theorem we have that the isomorphism classes of the category of $(V_{\mathcal{R}}, \eta)$-bundles over a connected manifold X with $\mathcal{R}^n$ as fiber are classified by the conjugation classes of representations of the form $\pi_1 \to GL(n, \mathcal{R})$, where π_1 is the fundamental group of the manifold X.

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